

A Dirac type variant of the xp Model and the Riemann Zeros

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We propose a Dirac type modification of the xp -model to a $x \sigma \cdot p$ model on a semi-infinite cylinder. This model is inspired by recent work of Sierra et al on the xp -model on the half-line. Our model realizes the Berry-Keating conjecture on the Riemann zeros. We indicate the connection of our model to that of gapped graphene with a supercritical Coulomb charge, which might provide a physical system for the study of the zeros of the Riemann Zeta function.

There have been an increasing interest in physical models that might shed light on the Riemann hypothesis regarding the zeros of the Riemann Zeta functions. This program aims at a possible realization of a conjecture by Polya and Hilbert on the zeros of the Riemann Zeta function [1, 2]. The Polya-Hilbert conjecture [2] states that the complex zeros of the Riemann Zeta function on the critical line are described by the eigenvalues of a self-adjoint operator. One of the original proposals for such an operator arose from the work of Berry [3, 4], followed by Connes [5] and Sierra [6–11] with their collaborators. The attempts towards construction of such an operator are guided by various properties that are ascribed to it [3, 4], including the requirements that it should exhibit a chaotic classical dynamics with isolated periodic trajectories and that it should break time reversal invariance.

Guided by these requirements, Berry and Keating [4] suggested that the operator $H_0 = xp$ is a good candidate for the realization of the Polya-Hilbert conjecture. The Berry-Keating operator xp leads to chaotic classical dynamics and breaks time reversal invariance. However, it does not by itself lead to closed trajectories, which requires various additional identifications in the classical phase space. The Berry-Keating operator was also analyzed by A. Connes [5].

An interesting physical realization of the Berry-Keating operator was given by Sierra and Townsend [7], who observed that the xp -operator appears as the Hamiltonian of the quantum Hall effect when restricted to the lowest Landau level [12]. They further argued that from this perspective, the Berry-Keating Hamiltonian has a quantum description consistent with the Polya-Hilbert conjecture. The relationship of Riemann zeros to other quantum systems such as the inverted harmonic oscillator [13] and the Morse potential [14] have also been

discussed in the literature.

Recently, Sierra and Rodriguez-Laguna [10] have proposed a new operator $H_1 = x \left(p + \frac{l_p^2}{p} \right)$ where the particle is restricted to the (regularized) half-line $l_x \leq x \leq \infty$ and l_p^2 is a constant. The operator H_1 satisfies the essential properties of the Berry-Keating model and in addition gives rise to closed classical trajectories. In the quantum theory, the appearance of the $\frac{1}{p}$ term in H_1 leads to a nonlocal boundary condition. With an appropriate choice of the kernel of $\frac{1}{p}$, H_1 was shown to admit a 1-parameter family of self-adjoint extensions with a spectrum that is bounded from below and is thus a legitimate quantum Hamiltonian operator. For certain choices of the self-adjoint extension parameter, this model seems to be consistent with the Polya-Hilbert conjecture.

The relation of the Hamiltonian operators in [3–5, 7, 10, 11] to the Polya-Hilbert conjecture appears through the Riemann-van Mangoldt formula

$$N(E) = \langle N(E) \rangle + S(E), \quad (1)$$

where $N(E)$ is the number of complex zeros of the Riemann Zeta function with positive imaginary part less than E , $\langle N(E) \rangle$ is the smooth part of $N(E)$ and $S(E)$ denotes a fluctuation in the counting (also known as error function) given by

$$S(E) = \frac{1}{\pi} \arg \zeta \left(\frac{1}{2} + iE \right) = \mathcal{O}(\log(E)). \quad (2)$$

The smooth part of Eq. (1) can be written as

$$\langle N(E) \rangle = \frac{\vartheta(E)}{\pi} + 1, \quad (3)$$

where the phase $\vartheta(E)$ of the Riemann Zeta function on the critical line (also known as Riemann-Siegel function [2]) is given by

$$\vartheta(E) = \arg \Gamma \left(\frac{1}{4} + \frac{iE}{2} \right) - \frac{1}{2} E \log \pi. \quad (4)$$

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The Hamiltonian operators introduced in [3–5, 7, 10, 11] provide an estimate for the phase $\vartheta(E)$ of the Riemann Zeta function, which can be used in the Riemann-van Mangoldt formula (1) to make inferences about the zeros of the Riemann Zeta function.

In this Letter, following the spirit of the work of Sierra et al [10, 11], we propose a variant of the xp model on the half-line where the operator p is replaced by a suitable two dimensional Dirac type operator \not{p} . The aim is to obtain a proper self-adjoint xp -operator on the half-line.

Let us consider the Dirac operator

$$\not{p} \equiv \sigma \cdot p = \sigma_x p_x + \sigma_y p_y, \quad (5)$$

where $p_a = -i\partial_a$, with $a = x, y$, and σ_x, σ_y are Pauli matrices. Thus,

$$\not{p} = -i \begin{pmatrix} 0 & \partial_x - i\partial_y \\ \partial_x + i\partial_y & 0 \end{pmatrix}. \quad (6)$$

This operator acts on two-component column-vectors valued on a Hilbert space defined on a semi-infinite cylinder. The semi-infinite cylinder is described by $0 \leq x < \infty$ and the y -direction is a circle of radius R . There are a $U(1)$ worth of possible boundary conditions that guarantees conservation of probability. This statement is equivalent to say that there is a $U(1)$ family of self-adjoint operators \not{p} on the semi-infinite cylinder.

The above considerations on the operator \not{p} should be contrasted with the fact that there is no self-adjoint operator p on the half-line. This statement can be checked by computing the deficiency indexes n_{\pm} and then use the criteria discovered by Von Neumann [15, 16]. The deficiency indexes n_{\pm} are defined as the dimension of the space of square-integrable solutions of the equations $p\psi_{\pm} = \pm i\psi_{\pm}$ on the half-line. For the operator p on the half line, we have $n_+ = 1$ and $n_- = 0$. Thus, by von Neumann theorem, there is no self-adjoint operator p on the half-line. Equivalently, there is no boundary condition on half-line that guarantees conservation of probability.

We now consider the Hamiltonian is $H = xp$. This is our proposed modified version of the xp -model. This Hamiltonian acts on a suitable domain of the Hilbert space of square-integrable functions on the cylinder. Let us consider the normal ordered operator¹

$$H = \sqrt{x} \not{p} \sqrt{x}. \quad (7)$$

Similar to the operator \not{p} , there is a $U(1)$ family of self-adjoint operators H on the semi-infinite cylinder. Indeed, H inherits the same possible self-adjoint domains from \not{p} . It may also noted that a different choice of normal ordering in the definition of H , like for instance $H =$

$(1/2)(xp + \not{p}x)$ does not change the conclusions of the following analysis.

For a two-component column-vector

$$\Psi(x, y) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (8)$$

the eigenvalue equation $H\Psi = E\Psi$ leads to

$$\sqrt{x}(\partial_x - i\partial_y)\sqrt{x}\psi_2 = iE\psi_1, \quad (9)$$

$$\sqrt{x}(\partial_x + i\partial_y)\sqrt{x}\psi_1 = iE\psi_2. \quad (10)$$

This system of equation is equivalent to a second order differential equation for one of the components, let us say $\psi_1(x, y)$. From the topology of the y -direction, we may consider the *ansatz*

$$\psi_1(x, y) = \frac{1}{\sqrt{2\pi}} \frac{\varphi(x)}{x} e^{i\frac{n+\alpha}{R}y}, \quad (11)$$

$$\psi_2(x, y) = \frac{1}{\sqrt{2\pi}} \frac{\chi(x)}{x} e^{i\frac{n+\alpha}{R}y}, \quad (12)$$

with $n \in \mathbb{Z}$ and $\alpha \in [0, 1]$. The parameter α is associated with the phase of the quasi-periodic boundary conditions along y -direction. It parameterizes the family of self-adjoint domains of the Hamiltonian. In more physical terms, α is related to a flux passing through the circle obtained by the compactification of the y -direction. If we now set

$$x = \frac{Ru}{2(n+\alpha)}, \quad (13)$$

we obtain

$$\frac{\partial^2 \varphi}{\partial u^2} + \left[-\frac{1}{4} + \frac{1}{2u} + \frac{E^2 + \frac{1}{4}}{u^2} \right] \varphi = 0, \quad (14)$$

which is exactly of the Whittaker's form [17]. It may be noted that the expression for ψ_2 may be obtained from that of ψ_1 .

In order to set up the allowed boundary conditions, we first regularize the semi-infinite line $0 \leq u < \infty$ to $u_0 \leq u < \infty$, for a positive u_0 . At u_0 we consider the boundary condition

$$\varphi(u_0) = 0. \quad (15)$$

Later we discuss the crucial limit $u_0 \rightarrow 0$. The analysis of boundary conditions presented below can be cast in a more formal language of self-adjoint domains and the von Neumann theorem [15, 16]. The boundary condition on the wave-functions for the singular attractive inverse-square potential following from the self-adjoint extension has been discussed in [18, 19]. The close relation between the boundary conditions following from the introduction of a cut-off to that obtained from self-adjoint extension has also been discussed in the literature [16, 19, 20]. Here we prefer to report on this route of regularization and renormalization, since it is more familiar among physicists.

¹ In the final stages of this work, we learned that a very similar model to this one was already suggested by M. Asorey, J. Esteve and G. Sierra on unpublished notes. Their motivations was the same as ours.

The general solution of Whittaker equation with boundary condition (15) is

$$\varphi(u) = A \left[W_{\frac{1}{2}, -iE}(u_0) W_{\frac{1}{2}, +iE}(u) - W_{\frac{1}{2}, +iE}(u_0) W_{\frac{1}{2}, -iE}(u) \right], \quad (16)$$

where $W(x)$ is Whittaker function

$$W_{k,m}(u) = e^{-\frac{u}{2}} u^{m+\frac{1}{2}} M \left(m - k + \frac{1}{2}, 1 + 2m; u \right),$$

with $M(a, b; u)$ being Kummer's confluent hypergeometric functions [17].

We would like to have a solution that is square-integrable. For that, we first note that as $u \rightarrow \infty$ [17],

$$M(a, b; u) \approx \frac{e^{i\pi a} \Gamma(b)}{\Gamma(b-a)} u^{-a} + \frac{\Gamma(b)}{\Gamma(a)} e^u u^{a-b} + \mathcal{O}\left(\frac{1}{u}\right).$$

Therefore, as $u \rightarrow \infty$,

$$\begin{aligned} \varphi(u) &\rightarrow Ae^{\frac{u}{2}} \frac{1}{\sqrt{u}} \left[W_{\frac{1}{2}, -iE}(u_0) \frac{\Gamma(1+2iE)}{\Gamma(+iE)} \right. \\ &\quad \left. - W_{\frac{1}{2}, +iE}(u_0) \frac{\Gamma(1-2iE)}{\Gamma(-iE)} \right] \end{aligned} \quad (17)$$

Thus the requirement of square-integrability implies the condition

$$\frac{W_{\frac{1}{2}, -iE}(u_0)}{W_{\frac{1}{2}, +iE}(u_0)} = \frac{\Gamma(1-2iE)}{\Gamma(1+2iE)} \frac{\Gamma(+iE)}{\Gamma(-iE)}. \quad (18)$$

In our setup, the quantity u_0 is a cut-off, which we would ideally like to take it to zero. Due to quantum instability associated with a strongly attractive inverse square potential, we cannot really remove it. As $u_0 \rightarrow 0$, we have [17]

$$W_{a, \pm iE}(u_0) \rightarrow u_0^{\frac{1}{2} \pm iE} \quad (19)$$

Using Eqs. (18) and (19), together with some basic properties of Gamma function [17], we get

$$\left(\frac{u_0}{8} \right)^{2iE} = - \frac{\Gamma\left(\frac{1}{4} + \frac{iE}{2}\right) \Gamma\left(\frac{3}{4} + \frac{iE}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{iE}{2}\right) \Gamma\left(\frac{3}{4} - \frac{iE}{2}\right)}. \quad (20)$$

Finally, using Eqs. (4) and (20), we get

$$e^{2i\vartheta(E) + iE \ln \pi} = - \left(\frac{u_0}{8} \right)^{2iE} \frac{\Gamma\left(\frac{3}{4} - \frac{iE}{2}\right)}{\Gamma\left(\frac{3}{4} + \frac{iE}{2}\right)}. \quad (21)$$

This is comparable (but not the same) with equation (21) of [7].

We have thus far shown that the $x\psi$ model together with the boundary condition (15) leads to an algebraic equation for the spectrum given by (21). On one side, this algebraic condition contains the phase of the Riemann-Zeta function (4) which is part of the Riemann-van Mangoldt counting formula (1). On the other side, apart from

the ratios of certain Gamma functions, it contains an undetermined parameter, namely the cut-off u_0 .

The appearance of the cut-off u_0 in (15) deserves some comments. As mentioned earlier, the cutoff u_0 is related to the self-adjoint extension parameter of the strongly attractive inverse square potential on the half-line [18, 19]. Similar to the discussions in [16, 20] of a conformal quantum mechanics, this cut-off implies a breaking of the scale symmetry by quantization. That means a scale anomaly emerges in this problem. Indeed, from a classical perspective, the $x\psi$ model, and also the $x\psi$, is scale invariant. Also from another perspective, the Hamiltonian associated with Eq. (14) in the short distance limit is singular and scale invariant. But the boundary condition (15) breaks this scale invariance. The analysis of such singular behavior for equations like Eq. (14) is given in [19]. The works [21–23] also contain important discussions for the emergence of scale anomaly in a similar context.

The restoration of the scale anomaly at the quantum level is relevant for the problem of Riemann Zeta function. Indeed, that allows us to enforce some of the requirements of Berry and Keating at the quantum level. One mechanism for the restoration of an otherwise anomalous symmetry is by the use of an appropriate mixed state [24]. This restoration mechanism in respect to the present problem is under current investigation and it will be reported elsewhere .

Another objective of this Letter is to present, following the spirit of the work of Sierra and Townsend [7], possible physical realizations for the $H = x\psi$ model.

To that end, let us first consider the Calogero model, whose solutions can be classified using the degree k of a polynomial that appears in the analysis [25]. It was shown in [26] that for $k = 0$, the Calogero model with a complex coupling has real eigenvalues. The corresponding differential equation (see Eq. (12) of [26]) has exactly the same form as Eq. (14), although the parameters appearing in the equations in these two cases have different interpretations. The main similarity between the two lies in the fact that they both describe a system of strongly coupled inverse square interaction. Such a system was first discussed by Landau, who observed that the strongly attractive inverse square coupling leads to a quantum instability that was characterized by “fall to the centre” [27]. Subsequently, a more complete quantum treatment was given by Case [19] who found that the spectrum is unbounded from below and is characterized by a one-parameter family of an undetermined constant. In technical terms, this unknown constant is nothing but the self- adjoint extension parameter, which encodes in itself the combined effects of the short distance physics. In this work, as well as in [26], the same role is played by the cut-off u_0 . Hence, the essential physics described here is that of a Calogero type system with a strongly attractive inverse square interaction.

This naturally leads to the question if there is indeed a physical system which realizes such a potential. If a physical system exists whose eigenvalue equation is ex-

actly governed by Eq. (14), then the spectrum of that physical system would provide a concrete realization of the Polya-Hilbert conjecture. While we do not have an exact answer to this question, but there is a graphene system whose equations are again very similar to what has been described above.

It is now well known that dynamics of low energy excitations in graphene is governed by a Dirac equation [28]. For our purpose, we consider the case of gapped graphene [29, 30]. In the presence of an external supercritical Coulomb impurity [30–33], the effective radial equation for gapped graphene [34] has the same formal structure as that of Eq. (14), although again with a different interpretation of the parameters. In addition, the solutions of a gapped graphene system with supercritical Coulomb charge [34] have the same form as Eq. (14) described above. When the external Coulomb charge in graphene is strongly attractive or equivalently in the supercritical region, the system again exhibits a quantum instability and “fall to the centre” [31–36], whose qualitative behaviour is similar to what we find here. In a graphene system, such an instability manifests itself through characteristic features in the local density of states (LDOS), which in principle can be measured using scanning tunneling microscope (STM) [31–33]. With suitable reinterpretation of the system parameters, the STM measurements of the LDOS in gapped graphene with a supercritical Coulomb charge may yield valuable information regarding the spectrum of our model considered here.

It may also be mentioned that the same equation obtained in [26], appears in the analysis of non-relativistic AdS/CFT correspondence [37]. Typically, such strongly attractive inverse square operators arise also in the analysis of near-horizon conformal structure of black holes [38, 39]. While none of these problems are identical to what has been considered here, the structural similarity of the equations and the corresponding solutions perhaps indicates a deeper relationship between these systems.

The systems described above share a common feature that they all correspond to a strongly attractive inverse square potential characterized by quantum instability. In such systems, it is possible to employ a renormalization group scheme to address the issue of the instability [40]. The instability indicates an incomplete understanding of the short distance physics. Typically such a system can be analyzed by first introducing a cut-off and then studying the renormalization group flow of the relevant parameters of the system, given by the corresponding β -function. It would be interesting to study the consequences of the RG flow for the system described here.

In summary, in this Letter, we have proposed a Dirac type variant of the $H = xp$ given in Eq. (7). We then analysed its spectrum and related it with the Riemann-van Mangoldt counting formula for the complex zeros of zeta function [2]. An important observation is that in order to find this quantum spectrum the scale invariance of the classical xp -model is necessarily broken. Based on this fact, we believe that the restoration of the scale invariance at the quantum level will have crucial implications to the Hilbert-Polya conjecture. Finally we suggested a physical realization of the present model in terms of the dynamics of the low energy excitations of the graphene in the presence of an external supercritical Coulomb impurity.

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